Another example on energy minimization model

(from past paper)

Given a noisy image I, consider the following denoising model:

$$E(f) = \int_D K_1(x,y)\left(f(x,y) - I(x,y)\right)^2 + \int_D K_2(x,y)\|\nabla f(x,y)\|^2,$$

where $K_1(x,y)$ and $K_2(x,y)$ are positive functions.

Please derive a PDE that must be satisfied by the minimizer of $E(f)$, and propose an iterative scheme to solve it.

Answer:

$$\left.\frac{d}{dt}\right|_{t=0} E(f+t\varphi) = \left.\frac{d}{dt}\right|_{t=0}\Big(\int_D K_1(x,y)\left(f(x,y)+t\varphi(x,y) - I(x,y)\right)^2$$

$$+ \int_D K_2(x,y)\|\nabla f(x,y) + t\nabla\varphi(x,y)\|^2\Big)$$

$$= \int_D \left.\frac{d}{dt}\right|_{t=0} K_1(x,y)\left(f(x,y)+t\varphi(x,y) - I(x,y)\right)^2$$

$$+ \int_D \left.\frac{d}{dt}\right|_{t=0} K_2(x,y)\|\nabla f(x,y) + t\nabla\varphi(x,y)\|^2$$

$$= \int_D 2K_1(x,y)\varphi(x,y)\left(f(x,y) + t\varphi(x,y) - I(x,y)\right)\Big|_{t=0}$$

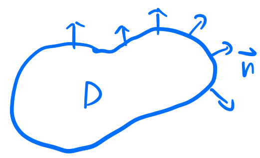
$$+ \int_D K_2(x,y)\left(2t\langle\nabla\varphi,\nabla\varphi\rangle + 2\langle\nabla f,\nabla\varphi\rangle\right)\Big|_{t=0}$$

$$= \int_D 2K_1(x,y)\varphi(x,y)\left(f(x,y) - I(x,y)\right)$$

$$+ \int_D 2K_2(x,y)\nabla f\cdot\nabla\varphi$$

(Divergence theorem:

$$\int_D \nabla\cdot\vec{F}\,dx = \int_{\partial D} \vec{F}\cdot\vec{n}\,d\sigma$$



)

$$= \int_D 2K_1(x,y)\,\varphi(x,y)\left(f(x,y) - I(x,y)\right)$$

$$\nabla\cdot(f\nabla g) = \nabla f\cdot\nabla g + f\nabla\cdot(\nabla g)$$

$$\nabla\cdot(f\vec{G}) = \nabla f\cdot\vec{G} + f\nabla\cdot(\vec{G})$$

$$+ 2\int_D \nabla\cdot\left(\varphi K_2\nabla f\right)(x,y) - 2\int_D \varphi(x,y)\,\nabla\cdot\left(K_2\nabla f\right)(x,y)$$

$$= 2\int_D \varphi(x,y)\,K_1(x,y)\left(f(x,y) - I(x,y)\right)$$

$$+ 2\int_{\partial D} \varphi(x,y)\,K_2(x,y)\,\nabla f(x,y)\cdot\vec{n}(x,y) - 2\int_D \varphi(x,y)\,\nabla\cdot\left(K_2\nabla f\right)(x,y)$$

$$\left.\frac{d}{dt}\right|_{t=0} E(f+t\varphi) = 0 \quad \text{for any } \varphi$$

$$\Rightarrow \begin{cases} -\nabla\cdot\left(K_2\nabla f\right)(x,y) + K_1(x,y)f(x,y) = K_1(x,y)\,I(x,y) & \text{on } D \\ \nabla f(x,y)\cdot\vec{n}(x,y) = 0 & \text{on } \partial D \end{cases}$$

How to minimize $E(f)$?

ignore this term.

Since $\frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} E(f+t\varphi) = \cancel{\int_{\partial D} \varphi(x,y)\,K_2(x,y)\,\nabla f(x,y)\cdot\vec{n}(x,y)}$

$$+ \int_I \varphi(x,y)\left(K_1(x,y)f(x,y) - K_1(x,y)\,I(x,y) - \nabla\cdot(K_2\nabla f)(x,y)\right)$$

So, in each iteration,

we update $f^{(k)}$ to $f^{(k+1)}$ via

$$f^{(k+1)} = f^{(k)} + t\left(K_1 I + \nabla\cdot\left(K_2\nabla f^{(k)}\right) - K_1 f^{(k)}\right)$$

Numerically, discretize it by any proper schemes.

Active Contour model

In denoise / deblur model, we aim to find a clean image f from noisy / blurred image I.

In the active contour model, given a image $I: \Omega \to \mathbb{R}$ we aim to find a curve $\gamma: [0, 2\pi] \longrightarrow \Omega$, s.t., γ segments an object in the image.

How to know whether a point in the image I is a boundary point of an object?

we need an edge detector function $V: \Omega \to \mathbb{R}$.

Then, we consider the following energy:

$$E_{snake}(\gamma) = \underbrace{\int_0^{2\pi} \frac{1}{2} |\gamma'(s)|^2 \, ds}_{\text{smothness term}} + \beta \underbrace{\int_0^{2\pi} V(\gamma(s)) \, ds}_{\text{fidelity term}}$$

We need to know how to minimize this energy w.r.t all possible γ.

A very simple example:

$$E(\gamma) = \int_0^{2\pi} \left(\|\gamma'(s)\|^2 + \lambda \|\gamma(s)\|^2 \right) ds$$

Suppose $\gamma \in C^2([0, 2\pi], \Omega)$. Then for arbitrary $\varphi \in C^2([0, 2\pi], \Omega)$

$$\left.\frac{d}{dt}\right|_{t=0} E(\gamma + t\varphi) = \left.\frac{d}{dt}\right|_{t=0} \int_0^{2\pi} \left(\|\gamma'(s) + t\varphi'(s)\|^2 + \lambda \|\gamma(s) + t\varphi(s)\|^2 \right) ds$$

$$= \int_0^{2\pi} \frac{d}{dt}\Big|_{t=0} \Big(\|\gamma'(s)\|^2 + t^2 \|\varphi'(s)\|^2 + 2t \langle \gamma'(s), \varphi'(s) \rangle$$
$$+ \lambda \|\gamma(s)\|^2 + \lambda t^2 \|\varphi(s)\|^2 + 2\lambda t \langle \gamma(s), \varphi(s) \rangle \Big) ds$$

$$= 2\int_0^{2\pi} \lambda \langle \gamma(s), \varphi(s) \rangle + \langle \gamma'(s), \varphi'(s) \rangle ds \quad \Big\downarrow (*).$$

$$= 2 \int_0^{2\pi} \langle \lambda \gamma(s) - \gamma''(s), \varphi(s) \rangle ds + 2 \langle \gamma'(s), \varphi(s) \rangle \Big|_{s=0}^{2\pi}$$

$$= 2 \int_0^{2\pi} \langle \lambda \gamma(s) - \gamma''(s), \varphi(s) \rangle ds$$

$\frac{d}{dt}\Big|_{t=0} E(\gamma + t\varphi) = 0$ for arbitrary φ

$$\Rightarrow \gamma''(s) - \lambda \gamma(s) = 0$$

In $(*)$, we use integration by parts:

for example, let $f, g : \mathbb{R} \to \mathbb{R}^2$.

write $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$

Then, $\int_0^a \langle f', g' \rangle dx = \int_0^a f_1' g_1' + f_2' g_2' \, dx$

$$= \int_0^a f_1' \, dg_1 + \int f_2' \, dg_2$$
$$= f_1' g_1 \Big|_0^a - \int_0^a g_1 f_1'' dx + f_2' g_2 \Big|_0^a - \int_0^a g_2 f_2'' dx$$

$$= \langle f', g \rangle \Big|_0^a - \int_0^a \langle g, f'' \rangle dx$$